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Compact flat manifolds with non-vanishing Stiefel–Whitney classes

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Abstract

We construct a class of compact flat Riemannian manifolds M of dimension $2n + 1$ with the following properties:

- (1) M has holonomy group $(\mathbb{Z}_2)^{n+1}$,
- (2) M is not a (non-trivial) flat toral extension of a compact flat manifold,
- (3) the first Betti number of M is 0, and
- (4) Stiefel–Whitney classes $w_{2j}(M)$ are non-zero for $0 \leq 2j \leq n$.

This is in the spirit of Vasquez's second example which is in error (Vasquez, 1970).

For each finite group Φ , there is a positive integer $N(\Phi)$ such that every flat manifold with holonomy group Φ has dimension higher than $N(\Phi)$, then M must be a non-trivial flat toral extension of a compact flat manifold. Vasquez pointed out that $N((\mathbb{Z}_2)^n) \geq n$ or $n - 1$ depending on n being even or odd. Our result shows that $N((\mathbb{Z}_2)^n) > 2n - 1$, a much sharper result. © 1999 Elsevier Science B.V. All rights reserved.

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There was the question whether all flat Riemannian manifolds are boundaries. As is well known, a flat manifold M is a boundary if and only if all of its Stiefel–Whitney numbers vanish. By [3], we know all flat manifolds are boundaries. (It was proved by analyzing the fixed point set of a finite group action.) Consequently, all Stiefel–Whitney numbers of a flat manifold vanish.

Before [3], people tried to solve the bounding problem by showing that all Stiefel–Whitney numbers of a flat manifold are 0, and suspected even the Stiefel–Whitney classes vanish. Auslander–Szczarba [1] was the first who found examples of flat manifolds, some

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of whose Stiefel–Whitney classes do not vanish. Later, Vasquez [6] constructed more such examples in relation to “toral extensions”. Flat manifolds which are toral extensions are easily seen to be boundaries. Vasquez’s first example is a torus bundle over a flat manifold, and all the even dimensional Stiefel–Whitney classes do not vanish up to the middle dimension.

The spirit of the second example in [6] was to find a class of flat manifolds each of which was not a torus extension (i.e., a toral bundle over another flat manifold), and having some non-vanishing Stiefel–Whitney classes. However, the groups in these examples turn out to contain non-trivial torsion (elements of finite order), hence they could not be the fundamental groups of flat manifolds.

We have found a class of $(2n + 1)$ -dimensional compact flat manifolds whose Stiefel–Whitney classes w_{2j} are non-zero for $0 \leq 2j \leq n$. Moreover, each one is not flat toral extension of another flat manifold.

It should be possible to prove the bounding problem for flat manifolds by directly showing vanishing of all Stiefel–Whitney numbers (cohomology calculations), not by 2-group actions on such manifolds (analyzing fixed points set and normal bundles). We hope that our examples shed some light in this direction.

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Let $E(m) = \mathbb{R}^m \rtimes O(m)$ be the group of isometries of \mathbb{R}^m . It has the group law

$$(a, A)(b, B) = (a + Ab, AB)$$

and acts on \mathbb{R}^m by

$$(a, A)x = a + Ax$$

for all $(a, A) \in E(m)$, $x \in \mathbb{R}^m$. We call a and A the translation part and rotation part of (a, A) , respectively. The subgroup of orientation-preserving isometries is $E_0(m) = \mathbb{R}^m \rtimes SO(m)$. A discrete subgroup π of $E(m)$ acting properly discontinuously on \mathbb{R}^m with compact quotient is called a crystallographic group. If, in addition, π acts freely (or equivalently, π is torsion-free), then π is called a Bieberbach group. In this case, the quotient \mathbb{R}^m/π is a manifold with fundamental group π . It is well known that a compact Riemannian manifold M is flat if and only if $M = \mathbb{R}^m/\pi$, where π is a Bieberbach group.

Bieberbach’s first theorem says that the fundamental group $\pi = \pi_1(M)$ of a compact flat Riemannian manifold M contains a normal subgroup \mathbb{Z}^m ($m = \text{dimension of } M$), of finite index. More precisely, let π be a cocompact discrete subgroup of $E(m) = \mathbb{R}^m \rtimes O(m)$. Then, $\pi \cap \mathbb{R}^m \cong \mathbb{Z}^m$ is a lattice of \mathbb{R}^m , and $\Phi = \pi/\mathbb{Z}^m$ is a finite holonomy group so that

$$1 \rightarrow \mathbb{Z}^m \rightarrow \pi \rightarrow \Phi \rightarrow 1$$

is exact. In fact, \mathbb{Z}^m is the unique maximal Abelian normal subgroup of π .

A *flat toral extension* of a compact flat Riemannian manifold was introduced in [6]. Let $Q \subset E(n)$ be a Bieberbach group so that it acts on \mathbb{R}^n freely and properly discontinuously. Suppose Q also acts on a torus T^k as isometries so that it acts on the product $T^k \times \mathbb{R}^n$ diagonally as isometries. The quotient $M = (T^k \times \mathbb{R}^n)/Q$ is called a flat toral extension

of the compact flat Riemannian manifold $N = \mathbb{R}^n/Q$. Then M is a torus bundle over N in the sense of Steenrod [5]. The structure group is Q acting on T^k as isometries. The total space is again a flat Riemannian manifold in an obvious way.

Let M be a flat manifold. Therefore, $\pi = \pi_1(M) \subset E(m)$ is a Bieberbach group. Suppose M is a flat toral extension of $N = \mathbb{R}^{m-k}/Q$ by a flat torus T^k . Then $\pi = \pi_1((T^k \times \mathbb{R}^{m-k})/Q)$ fits the short exact sequence

$$1 \rightarrow \pi_1(T^k) \rightarrow \pi \rightarrow Q \rightarrow 1.$$

Further, Q is torsion free since Q acts on \mathbb{R}^{m-k} freely. Conversely, suppose π contains a normal subgroup $A \subset \mathbb{Z}^m$ so that

$$1 \rightarrow A \rightarrow \pi \rightarrow Q \rightarrow 1$$

is exact, and Q is torsion free. Then A is isomorphic to \mathbb{Z}^k for some k , and Q embeds into $E(m-k)$ as a cocompact discrete subgroup so that M is an extension of a flat manifold $N = \mathbb{R}^{m-k}/Q$ by a torus $T^k = \mathbb{R}^k/A$. When $k = 0$ (i.e., A is trivial), the flat toral extension is called trivial.

Therefore, a flat manifold $M = \mathbb{R}^m/\pi$ is a flat toral extension if and only if π fits the following diagram

$$1 \rightarrow A \rightarrow \pi \rightarrow Q \rightarrow 1,$$

where $A \subset \mathbb{Z}^m$ and Q is torsion free.

Vasquez's first example. For each $n \geq 1$, let G_n be the subgroup of $\mathbb{R}^{2n+1} \rtimes O(2n+1)$ generated by $2n+1$ elements $t_1, \dots, t_{n+1}, \tau_1, \dots, \tau_n$, where t_i ($1 \leq i \leq n+1$) is a translation by one-unit along the i th axis; and τ_j ($1 \leq j \leq n$) translates one half unit along the $(n+1+j)$ th axis and simultaneously reflects in the $(j, j+1)$ plane. In coordinate notation,

$$\begin{aligned} t_i(x_1, \dots, x_{n+1}, y_1, \dots, y_n) \\ &= (x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_{n+1}, y_1, \dots, y_n), \\ \tau_j(x_1, \dots, x_{n+1}, y_1, \dots, y_n) \\ &= (x_1, \dots, x_{j-1}, -x_j, -x_{j+1}, x_{j+2}, \dots, x_{n+1}, y_1, \dots, y_j + 1/2, \dots, y_n). \end{aligned}$$

Then G_n is torsion-free Bieberbach group embedded in such a way that $M^{2n+1} = \mathbb{R}^{2n+1}/G_n$ is a compact flat Riemannian manifold. And we have a (split) exact sequence as follows:

$$0 \rightarrow \mathbb{Z}^{n+1} \rightarrow G_n \rightarrow \mathbb{Z}^n \rightarrow 1.$$

Note that M^{2n+1} is a non-trivial flat toral extension of a flat n -torus. And the Stiefel–Whitney classes $w_{2j}(M^{2n+1})$, $0 \leq 2j \leq n$, are non-zero; the others are 0 [6, Proposition 5.1].

Vasquez's second example. For each $i = 1, 2, \dots, 2n+1$, let $\sigma_i \in \mathbb{R}^{2n+1} \rtimes O(2n+1)$ be reflection in the i th axis, and let $s_i \in \mathbb{R}^{2n+1} \rtimes O(2n+1)$ be translation by one half unit in the i th axis. In coordinate notation,

$$\begin{aligned}\sigma_i(x_1, \dots, x_{2n+1}) &= (x_1, \dots, -x_i, \dots, x_{2n+1}), \\ s_i(x_1, \dots, x_{2n+1}) &= (x_1, \dots, x_i + 1/2, \dots, x_{2n+1}).\end{aligned}$$

Let $h_i = \sigma_i \sigma_{i+1} s_{(n+1)+i}$, $i = 1, 2, \dots, 2n+1$ (the subscripts are to be interpreted modulo $2n+1$), and let \overline{G}_{2n} be the subgroup of $\mathbb{R}^{2n+1} \rtimes O(2n+1)$ generated by h_1, \dots, h_{2n} , and t_{n+1} . Note that

$$h_i^2 = t_{(n+1)+i}, \quad 1 \leq i \leq 2n.$$

Therefore, the maximal normal Abelian subgroup is isomorphic to \mathbb{Z}^{2n+1} which is generated by $t_1 = h_{n+1}^2, \dots, t_n = h_{2n}^2, t_{n+1}, t_{n+2} = h_1^2, \dots, t_{2n+1} = h_n^2$. The quotient $\Phi = \overline{G}_{2n}/\mathbb{Z}^{2n+1}$ is $(\mathbb{Z}_2)^{2n}$ generated by the images of h_1, h_2, \dots, h_{2n} . The sequence of groups

$$1 \rightarrow \mathbb{Z}^{2n+1} \rightarrow \overline{G}_{2n} \rightarrow (\mathbb{Z}_2)^{2n} \rightarrow 1$$

is exact.

It is well known that \overline{G}_{2n} acts freely on \mathbb{R}^{2n+1} if and only if \overline{G}_{2n} is torsion free. However,

$$(h_i h_{n+i})^2 = 1 \quad \text{for all } 1 \leq i \leq 2n.$$

This shows that \overline{G}_{2n} does not act freely, and $\overline{M}^{2n+1} = \mathbb{R}^{2n+1}/\overline{G}_{2n}$ is not a manifold.

Theorem. *There exists a class of compact flat Riemannian manifolds M of dimension $2n+1$ with the following properties:*

- (1) M has holonomy group $(\mathbb{Z}_2)^{n+1}$,
- (2) M is not a (non-trivial) flat toral extension of a compact flat manifold,
- (3) the first Betti number of M is 0, and
- (4) Stiefel–Whitney classes $w_{2j}(M)$ are non-zero for $0 \leq 2j \leq n$.

Recall the definition of $N(\Phi)$ from [6]: *There is associated with each finite group Φ a positive integer $N(\Phi)$ such that: if M is a compact flat manifold with holonomy group Φ , and $\dim(M) > N(\Phi)$, then M is a flat toral extension of another flat manifold of dimension $\leq N(\Phi)$.* It is noted that [6, Corollary 5.3] $N((\mathbb{Z}_2)^{2n}) \geq 2n$ and $N((\mathbb{Z}_2)^{2n+1}) \geq 2n$. In other words, $N((\mathbb{Z}_2)^n) \geq n$ or $n-1$ depending on n being even or odd. Our example shows this number must be, in fact, much bigger.

Corollary. $N((\mathbb{Z}_2)^n) \geq 2n-1$.

Proof. We shall show $N((\mathbb{Z}_2)^{n+1}) \geq 2n+1$. Suppose otherwise. Since our manifold M in theorem has dimension $2n+1$, we have $\dim(M) > N((\mathbb{Z}_2)^{n+1})$. Thus M must be a non-trivial flat toral extension of another flat manifold. But, this is impossible by (2) of the statement in theorem. \square

Proof of Theorem. The rest of this paper will be occupied by a proof of the theorem. For each $n \geq 1$, let π_n be the subgroup of $\mathbb{R}^{2n+1} \rtimes O(2n+1)$ generated by $t_1, \dots, t_{n+1}, \tau_1, \dots, \tau_n$ and K , where

- t_i ($1 \leq i \leq n+1$) translates one unit along the i th axis,

- τ_j ($1 \leq j \leq n$) reflects the j th and $(j+1)$ st axes and simultaneously translates one half unit along the $(n+1+j)$ th axis,
- K reflects the last $n+1$ axes when n is odd (the last n axes when n is even) and simultaneously translates one half unit along each of the first $n+1$ axes.

In coordinate notation,

$$\begin{aligned}
 t_i(x_1, \dots, x_{n+1}, y_1, \dots, y_n) &= (x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_{n+1}, y_1, \dots, y_n), \quad 1 \leq i \leq n+1; \\
 \tau_j(x_1, \dots, x_{n+1}, y_1, \dots, y_n) &= (x_1, \dots, x_{j-1}, -x_j, -x_{j+1}, x_{j+2}, \dots, x_{n+1}, y_1, \dots, y_j + 1/2, \dots, y_n), \\
 &\quad 1 \leq j \leq n; \\
 K(x_1, \dots, x_{n+1}, y_1, \dots, y_n) &= \begin{cases} (x_1 + \frac{1}{2}, \dots, x_n + \frac{1}{2}, -x_{n+1} + \frac{1}{2}, -y_1, \dots, -y_n) & \text{if } n \text{ is odd,} \\ (x_1 + \frac{1}{2}, \dots, x_{n+1} + \frac{1}{2}, -y_1, \dots, -y_n) & \text{if } n \text{ is even.} \end{cases}
 \end{aligned}$$

Note that t_1, \dots, t_{n+1} and τ_1, \dots, τ_n are the same as in Vasquez's first example G_n . In fact, G_n is a subgroup of π_n , of index 2. This is obvious since $K^2 = t_1 t_2 \cdots t_{n+1}$ or $K^2 = t_1 t_2 \cdots t_n$ depending on where n is even or odd. Also note that

$$\tau_j^2 = t_{(n+1)+j} \quad (1 \leq j \leq n)$$

is the translation one unit along the $((n+1)+j)$ th axis.

A note on the indices: To denote the first $(n+1)$ coordinates, we use $i = 1$ to $n+1$. To denote the last n coordinates, we use $j = 1$ to n .

From the presentation of π_n , it is easy to see that π_n has holonomy group $(\mathbb{Z}_2)^{n+1}$, which shows (1). For a more precise argument, see the first part of proof of (4). (The first example G_n is a subgroup of π_n of index 2.) First we prove that π_n is torsion free and show (2)–(4) of the theorem.

π_n is torsion free: We first show that

$$G_n = \langle t_1, \dots, t_{n+1}, \tau_1, \dots, \tau_n \rangle$$

is torsion free. A general element of G_n is of the form

$$\prod_{i=1}^{n+1} t_i^{a_i} \cdot \prod_{j=1}^n t_{(n+1)+j}^{b_j} \cdot \prod_{j=1}^n \tau_j^{\varepsilon_j},$$

where $a_i, b_i \in \mathbb{Z}$; ε_j is 0 or 1. For convenience, let us write

$$S = \prod_{i=1}^{n+1} t_i^{a_i}, \quad T = \prod_{j=1}^n t_{(n+1)+j}^{b_j}, \quad U = \prod_{j=1}^n \tau_j^{\varepsilon_j}.$$

Clearly $ST \in \mathbb{Z}^{2n+1}$ is not a torsion. We show STU is not a torsion. Observe that the matrix part of U is a diagonal matrix whose entries are ± 1 's,

$$\mathcal{D}_U = \{(-1)^{\varepsilon_1}, (-1)^{\varepsilon_1+\varepsilon_2}, (-1)^{\varepsilon_2+\varepsilon_3}, \dots, (-1)^{\varepsilon_{n-1}+\varepsilon_n}, (-1)^{\varepsilon_n}, 1, 1, \dots, 1\},$$

where the number of 1's at the end is n . One should note that there are always even number of -1 's among this set, because a non-zero ε_j contributes two -1 's. We partition the set $\{1, 2, 3, \dots, n, n+1\}$ into two subsets:

$$P = \{1 \leq i \leq n+1 \mid i\text{th entry of } \mathcal{D}_U \text{ is } +1\},$$

$$N = \{1 \leq i \leq n+1 \mid i\text{th entry of } \mathcal{D}_U \text{ is } -1\}.$$

Then N has even number of elements. Clearly,

$$USU^{-1} = \prod_{i \in P} t_i^{a_i} \prod_{i \in N} t_i^{-a_i}.$$

So, conjugation by U changes

$$S = \prod_{i \in P} t_i^{a_i} \prod_{i \in N} t_i^{a_i} \quad \text{to} \quad \widehat{S} = \prod_{i \in P} t_i^{a_i} \prod_{i \in N} t_i^{-a_i}.$$

From

$$US = \widehat{S}U, \quad UT = TU,$$

we get

$$(STU)^2 = STU STU = ST\widehat{S}UTU = ST\widehat{S}TU^2 = S\widehat{S}T^2U^2 \in \mathbb{Z}^{2n+1}.$$

An easy calculation shows

$$S\widehat{S} = \prod_{i \in P} t_i^{2a_i}, \quad T^2 = \prod_{j=1}^n t_{(n+1)+j}^{2b_j}, \quad U^2 = \prod_{j=1}^n t_{(n+1)+j}^{\varepsilon_j}.$$

Moreover,

$$S\widehat{S} \in \langle t_1, \dots, t_{n+1} \rangle \quad \text{and} \quad T^2U^2 \in \langle t_{n+2}, \dots, t_{2n+1} \rangle.$$

Suppose STU has finite order. Then $S\widehat{S} = 1$ and $T^2U^2 = 1$. The latter is equivalent to $b_j = 0, \varepsilon_j = 0$ for all $j = 1, \dots, n$. If this happens, then $TU = 1$ so that $STU = S$, which also must be the identity. We have shown STU is not of finite order unless it is the identity element. This proves G_n is torsion free.

Since $\pi_n/G_n = \mathbb{Z}_2$ and G_n is torsion free, if π_n has a torsion, it must have order 2, which is of the form $STUK$.

First suppose n is even. Then

$$KS = SK, \quad KT = T^{-1}K, \quad KU = \prod_{i \in N} t_i \prod_{j=1}^n t_{(n+1)+j}^{-\varepsilon_j} UK.$$

We calculate $(STUK)^2$,

$$\begin{aligned} (STUK)^2 &= STUST^{-1}KUK \\ &= ST\widehat{S}T^{-1}U(KU)K \\ &= ST\widehat{S}T^{-1}U \cdot \prod_{i \in N} t_i \prod_{j=1}^n t_{(n+1)+j}^{-\varepsilon_j} \cdot UK^2 \end{aligned}$$

$$\begin{aligned}
&= S\hat{S} \prod_{i \in N} t_i^{-1} \prod_{j=1}^n t_{(n+1)+j}^{-\varepsilon_j} U^2 K^2 \\
&= \prod_{i \in P} t_i^{2a_i} \prod_{i \in N} t_i^{-1} K^2 \quad \text{since } U^2 = \prod_{j=1}^n t_{(n+1)+j}^{\varepsilon_j} \\
&= \prod_{i \in P} t_i^{2a_i} \prod_{i \in N} t_i^{-1} \prod_{i=1}^{n+1} t_i \\
&= \prod_{i \in P} t_i^{2a_i} \prod_{i \in P} t_i \\
&= \prod_{i \in P} t_i^{2a_i+1}.
\end{aligned}$$

Since n and $|N|$ are even, $|P| + |N| = n + 1$ shows that $|P|$ is odd. Therefore, P is non-empty, and the last term can never be the identity element. We have shown that $(STUK)^2 \neq 1$.

Now suppose n is odd. For simplicity, we write $t_{n+1}^{a_{n+1}}$ by s . Then

$$KS = Ss^{-2}K, \quad KT = T^{-1}K, \quad KU = \prod_{i \in N} t_i \prod_{j=1}^n t_{(n+1)+j}^{-\varepsilon_j} UK.$$

We calculate $(STUK)^2$.

$$\begin{aligned}
(STUK)^2 &= STUSs^{-2}T^{-1}KUK \\
&= ST\hat{S}\hat{s}^{-2}T^{-1}U(KU)K, \quad \hat{s} = s^{\pm 1} \text{ depending on } n+1 \in P \text{ or not} \\
&= ST\hat{S}\hat{s}^{-2}T^{-1}U \cdot \prod_{i \in N} t_i \prod_{j=1}^n t_{(n+1)+j}^{-\varepsilon_j} \cdot UK^2 \\
&= S\hat{S}\hat{s}^{-2} \prod_{i \in N} t_i^{-1} \prod_{j=1}^n t_{(n+1)+j}^{-\varepsilon_j} U^2 K^2 \\
&= \prod_{i \in P} t_i^{2a_i} \hat{s}^{-2} \prod_{i \in N} t_i^{-1} K^2 \quad \text{since } U^2 = \prod_{j=1}^n t_{(n+1)+j}^{\varepsilon_j} \\
&= \prod_{i \in P} t_i^{2a_i} \hat{s}^{-2} \prod_{i \in N} t_i^{-1} \prod_{i=1}^n t_i \\
&= \prod_{i \in P} t_i^{2a_i} \hat{s}^{-2} \prod_{i \in N} t_i^{-1} \prod_{i=1}^{n+1} t_i \cdot t_{(n+1)+1}^{-1} \\
&= \left(\prod_{i \in P} t_i^{2a_i} \prod_{i \in P} t_i \right) (\hat{s}^{-2} t_{(n+1)+1}^{-1}) \\
&= \prod_{i \in P} t_i^{2a_i+1} \cdot t_{n+1}^{\mp 2a_{n+1}-1}, \quad \text{depending on } n+1 \in P \text{ or not.}
\end{aligned}$$

Since n is odd and $|N|$ is even, $|P| + |N| = n + 1$ shows that $|P|$ is even. When $n + 1 \in P$, there are two factors of powers of t_{n+1} in the last term which are canceled. Therefore, in this case, the last term contains $|P| - 1$ distinct factors. When $n + 1 \notin P$, the last term has $|P| + 1$ distinct factors. In any case, $|P| \pm 1$ is odd, and the last term can never be the identity element. We have shown that $(STUK)^2 \neq 1$. We conclude that π_n is torsion free.

Now, we obtain a compact flat Riemannian manifold $M = \mathbb{R}^{2n+1}/\pi_n$.

(2) Suppose M is a non-trivial flat toral extension of a compact flat Riemannian manifold. Then there is an exact sequence

$$1 \rightarrow A \rightarrow \pi_n \rightarrow Q \rightarrow 1,$$

where $A \subset \mathbb{Z}^{2n+1}$ is non-trivial and Q is torsion free. Let

$$ST = \prod_{i=1}^{n+1} t_i^{a_i} \cdot \prod_{j=1}^n t_{(n+1)+j}^{b_j} \in A.$$

Then $KSTK^{-1} \in A$ since A is normal in π_n .

Claim (a). $T = 1$; that is, $b_j = 0$ for $j = 1, \dots, n$.

(i) When n is even. Since $KSTK^{-1} = ST^{-1}$ and $t_{(n+1)+j} = \tau_j^2$,

$$(KSTK^{-1})(ST)^{-1} = T^{-2} = \prod_{j=1}^n t_{(n+1)+j}^{-2b_j} = \prod_{j=1}^n \tau_j^{-4b_j}.$$

Since A is normal in π_n and $ST \in A$, $K(ST)K^{-1}(ST)^{-1} \in A$. Thus,

$$\prod_{j=1}^n \tau_j^{-4b_j} \in A.$$

If all b_j 's are 0, then we are done. Otherwise, let p be the greatest common divisor of $\{b_1, b_2, \dots, b_n\}$. Then

$$\left(\prod_{j=1}^n \tau_j^{b'_j} \right)^{-4p} \in A,$$

where $b_j = pb'_j$ for each j . Since one of b_j 's is odd, $\prod_{j=1}^n \tau_j^{b'_j} \notin A$. We have found an element of π_n which is not in A , but its $4p$ th power lies in A . Such an element becomes a torsion element of Q . Thus, for Q to be torsion free, all b_j 's must be 0.

(ii) When n is odd. Instead of K in the case when n is even, now we use τ given by $\tau = \tau_1 \tau_3 \cdots \tau_n$ (product of odd terms). Then the matrix part of τ has exactly the first $(n+1)$ of -1 's and all the rest are $+1$'s. Therefore,

$$\tau(ST)\tau^{-1} = S^{-1}T.$$

Then

$$\tau(ST)\tau^{-1}(ST) = T^2 = \prod_{j=1}^n t_{(n+1)+j}^{2b_j} = \prod_{j=1}^n \tau_j^{4b_j}.$$

Since A is normal in π_n and $ST \in A$, $\tau(ST)\tau^{-1}(ST) \in A$. Thus,

$$\prod_{j=1}^n \tau_j^{4b_j} \in A.$$

By the same argument as in the case of n even, we conclude that $b_j = 0$ for all j .

Claim (b). $S = 1$; namely, $a_i = 0$ for $i = 1, 2, \dots, n+1$.

By Claim (a), we have $S \in A$. We write the product of all τ_j 's by τ_{n+1} . Then the matrix part of τ_{n+1} has -1 's in the first and $(n+1)$ st slot, and all the rest $+1$'s.

For each $1 \leq i \leq n+1$, one can find a product u_i of elements of $\tau_1, \tau_2, \dots, \tau_n, \tau_{n+1}$ and K whose matrix part has diagonal entry $+1$ in the i th slot only and all the rest -1 's. More precisely, we can get (regardless of n being even or odd)

$$\begin{aligned} u_i(x_1, \dots, x_i, \dots, x_{n+1}, y_1, \dots, y_n) \\ = (-x_1 + 1/2, \dots, +x_i + 1/2, \dots, -x_{n+1} + 1/2, -y_1 + d_1, \dots, -y_n + d_n), \end{aligned}$$

where $d_j \in \mathbb{R}$ for each j . Then

$$t_i^{2a_i} = (u_i S u_i^{-1}) S \in A.$$

On the other hand, $u_i^2 = t_i$. Therefore, $u_i^{4a_i} = t_i^{2a_i} \in A$ and $u_i \notin A$. Therefore, if $a_i \neq 0$, then Q has a torsion generated by u_i . Consequently, all a_i 's must vanish.

Thus, if $ST \in A$, then $ST = 1$ so that A is the trivial group. We have shown that M is not a non-trivial flat toral extension of any compact flat Riemannian manifold.

(3) For each $1 \leq i \leq 2n+1$, there exists a v_i (which is some τ_j or K) where i th diagonal entry is -1 . Then $[t_i, v_i] = t_i v_i t_i^{-1} v_i^{-1} = t_i^2$. Therefore $t_1^2, t_2^2, \dots, t_{2n+1}^2 \in [\pi_n, \pi_n]$, and $H_1(\pi_n; \mathbb{Z}) = \pi_n / [\pi_n, \pi_n]$ is finite so that the first Betti number of M is 0.

(4) Let $M' = \mathbb{R}^{2n+1} / G_n$, the space in Vasquez's first example, and $M = \mathbb{R}^{2n+1} / \pi_n$. Recall the short exact sequence

$$1 \rightarrow G_n \rightarrow \pi_n \rightarrow \mathbb{Z}_2 \rightarrow 1,$$

where \mathbb{Z}_2 is generated by the image of K . Then our manifold M is doubly covered by M' . The covering projection $p: M' \rightarrow M$ induces a cohomology homomorphism

$$p^*: H^*(M; \mathbb{Z}_2) \rightarrow H^*(M'; \mathbb{Z}_2).$$

Let w_j, w'_j be the Stiefel–Whitney classes of M and M' , respectively. By the naturality of the Stiefel–Whitney classes, $p^*(w_j) = w'_j$. By Vasquez's result (first example), $w'_{2j} \neq 0$ for $1 \leq 2j \leq n$. Therefore, $w_{2j} \neq 0$ for $1 \leq 2j \leq n$. This completes the proof. \square

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